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Nonlinear Oscillations of a Controlled Periodic System

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The motion of a spinning symmetric satellite in an elliptic orbit is examined. The linearized equations of motion for the system have periodic coefficients, and a modal control scheme for such systems is used to stabilize the linearized equations. The nonlinear behavior of both the controlled and uncontrolled system is examined using Poincaré maps. The behavior of the nonlinear controlled system is shown to depend on the controller gain with stable periodic solutions appearing at higher gains.

Introduction

THE use of Floquet theory to study the stability of time-periodic systems has had wide application in the field of satellite attitude dynamics. Kane and Barba[1] examined the stability of a spinning symmetrical satellite in an elliptic orbit through the use of Floquet theory. More recently, Calico and Wiesel[2] developed a modal control method for linear time-periodic systems. Calico and Yeakel[3] as well as Calico et al.[4] have applied this method to the control of the attitude motion of a spinning symmetric satellite in an elliptic orbit.

The current work considers the nonlinear behavior of both the uncontrolled and controlled attitude motion of such a satellite system. A modal controller is described for the linear time-periodic system of equations representing the small perturbation motion of the satellite, and its performance as a function of controller gain is determined. The nonlinear behavior of this system is then considered by examining the phase planes associated with the controller modal variables.

System Dynamics

The equations of motion for a spinning symmetrical satellite in an elliptical orbit are derived in Ref. 1. The coordinate systems used are shown in Fig. 1. The present analysis uses the same dynamic model but adds nondimensional control torques m_1 and m_2 about the axes C_1 and C_2, respectively. With this addition, the equations of motion become

$$\begin{aligned}\theta_1'' &= 2\theta_1'\theta_2' \tan\theta_2 + 2\nu'\theta_2' \cos\theta_1 \\ &\quad - (K+1)(\alpha+1)(\theta_2' + \nu' \sin\theta_1)\sec\theta_2 + \nu'' \cos\theta_1 \tan\theta_2 \\ &\quad + \nu'\nu' \sin\theta_1 \cos\theta_1 + m_1 \sec\theta_2 \end{aligned} \tag{1}$$

$$\begin{aligned}\theta_2'' &= (K+1)(\alpha+1)\theta_1' \cos\theta_2 - \nu' \cos\theta_1 \sin\theta_2 - \nu'' \sin\theta_1 \\ &\quad - 2\nu'\theta_1' \cos\theta_1 \cos^2\theta_2 - (\theta_1'^2 - \nu'^2 \cos^2\theta_1 + 3K\zeta^{-3}) \\ &\quad \times \sin\theta_2 \cos\theta_2 + m_2 \end{aligned} \tag{2}$$

In Eqs. (1) and (2) the prime derivatives are with respect to nondimensional time τ defined as $\tau = nt$, where n is the mean motion of the satellite. The parameters K, ζ, ν', and α are all nondimensional quantities defined by

$$K = \frac{C-A}{A}, \quad \zeta = \frac{r}{a}, \quad \nu' = \frac{\dot\theta}{n}, \quad \alpha = \frac{\omega_3 - n}{n} \tag{3}$$

where C and A are the satellites' moments of inertia, a the length of the orbit semimajor axis, r the orbit radius, $\dot\theta$ the orbital rate, and ω_3 the constant angular velocity about the spin axis of the satellite.

Equations (1) and (2) may be numerically integrated once the values of K and α are chosen, m_1 and m_2 are specified, and ζ, ν', and ν'' are determined by solving the standard two-body orbit equation. The solutions to these equations represent the nonlinear behavior of the satellite.

Linearization

To perform a stability analysis and a subsequent control design, Eqs. (1) and (2) will be linearized. The equilibrium solution of interest is represented by the satellite spinning at a

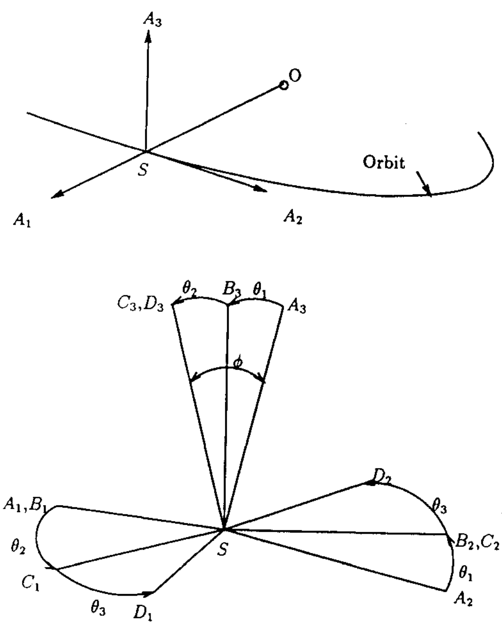

Fig. 1 Orbit reference frames.

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constant rate about the axis C_3 with this axis aligned with A_3. Hence, the angles θ_1 and θ_2 are both zero in equilibrium. Defining the state $\bar{x}^T = \{\theta_1, \theta_2, \theta_1', \theta_2'\}$, the linearized equations of motion can be written in the form

$$\bar{x}'(\tau) = A(\tau)\bar{x}(\tau) + B\bar{u}(\tau) \tag{4}$$

where the state matrix $A(\tau)$ is given by

$$A(\tau) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_{31} & a_{32} & 0 & a_{34} \\ a_{41} & a_{42} & a_{43} & 0 \end{bmatrix} \tag{5}$$

where

$$a_{31} = (1-\epsilon^2)\zeta^{-4} - (1+\alpha)(1+K)(1-\epsilon^2)^{1/2}\zeta^{-2}$$

$$a_{32} = -a_{41} = -2(1-\epsilon^2)^{1/2}\zeta'\zeta^{-3} - (1+\alpha)(1+K)$$

$$a_{34} = -a_{43} = 2(1-\epsilon^2)^{1/2}\zeta^{-2} - (1+\alpha)(1+K)$$

$$a_{42} = a_{31} - 3K\zeta^{-3}$$

The parameter ϵ is the orbital eccentricity. It should be noted that the elements of $A(\tau)$ are periodic with period T equal to the orbital period.

A scalar control has been assumed in Eq. (4), and the control is normalized such that the matrix B can be written in the form

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \tag{6}$$

Control Design

The system in Eq. (4) is a linear system with periodic coefficients. As such, it may be analyzed using Floquet theory. A fundamental result of Floquet theory is that the principal fun-

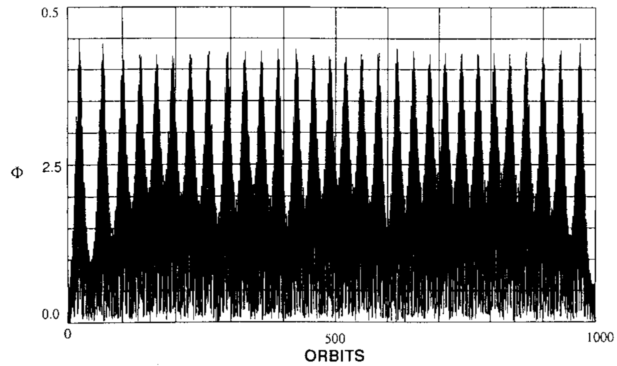

Fig. 2 Uncontrolled nonlinear ϕ response for 1000 orbits.

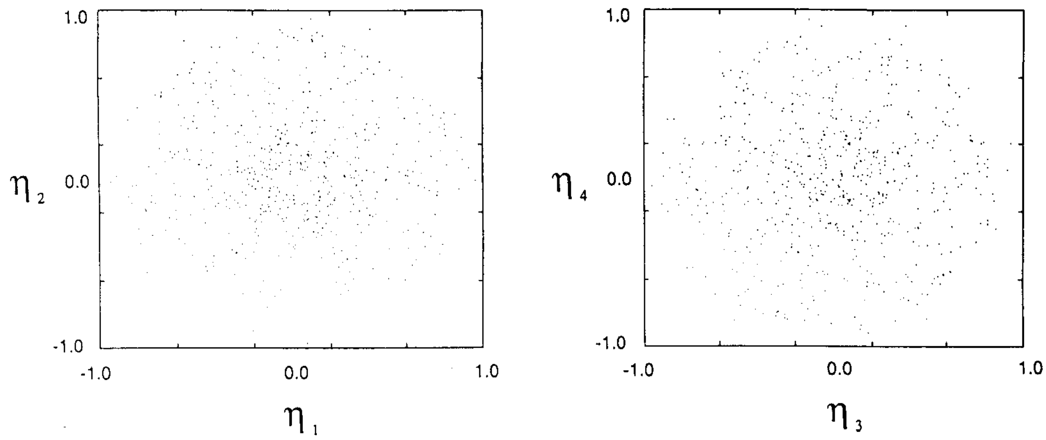

Fig. 3 Poincaré map or uncontrolled nonlinear system.

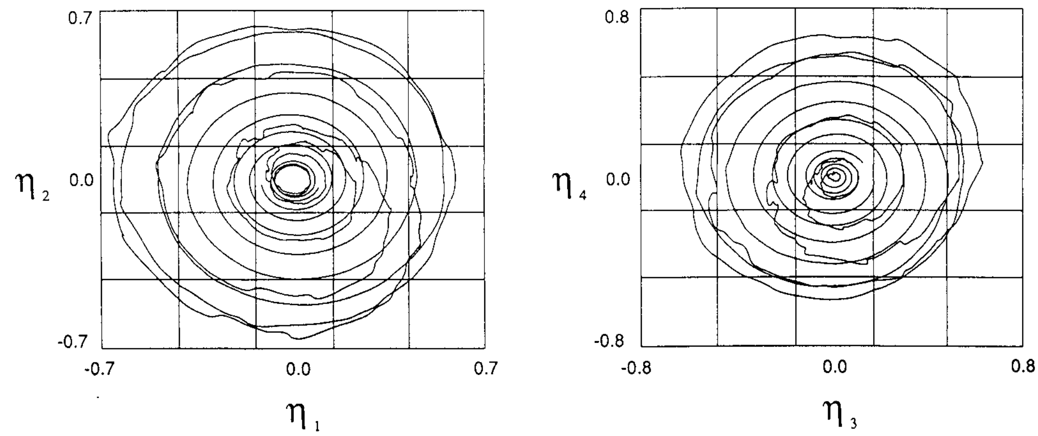

Fig. 4 Phase planes of uncontrolled nonlinear system.

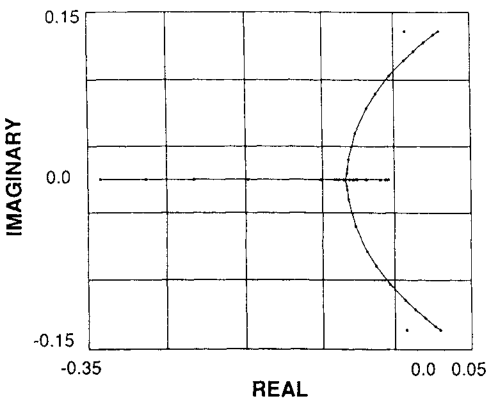

Fig. 5 Root locus of linearized system.

damental for the system (4) may be written in the form

$$\phi(\tau,0) = F(\tau)e^{J\tau}F^{-1}(0) \tag{7}$$

where $F(\tau)$ is a periodic modal matrix of period T, and J is a constant matrix. Since $F(\tau)$ is periodic, the stability of the system is determined by the eigenvalues of the matrix J. These eigenvalues may be determined by evaluating $\phi(\tau,0)$ at $\tau = T$, to obtain

$$\phi(T,0) = F(0)e^{J\tau}F^{-1}(0) \tag{8}$$

Using the modal matrix $F(\tau)$, we define coordinates $\bar{\eta}(\tau)$ such that

$$\bar{x}(\tau) = F(\tau)\bar{\eta}(\tau) \tag{9}$$

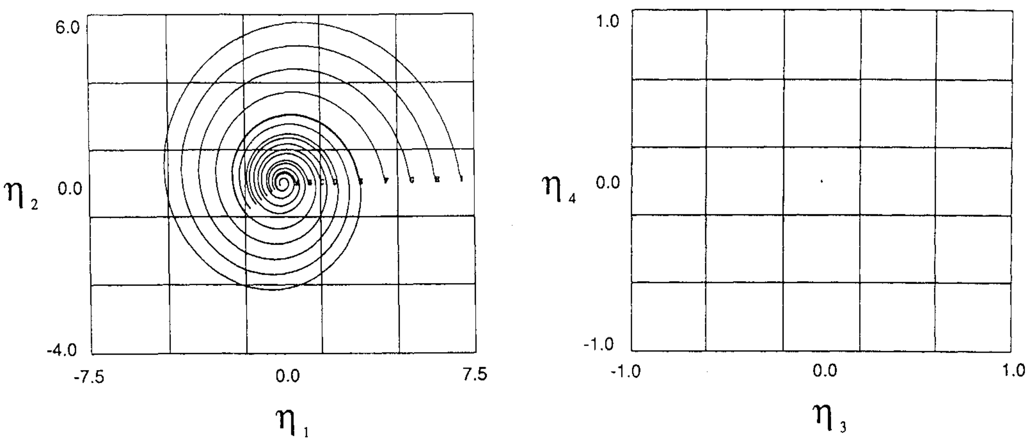

Fig. 6 Phase planes of controlled linearized system, gain = 0.4.

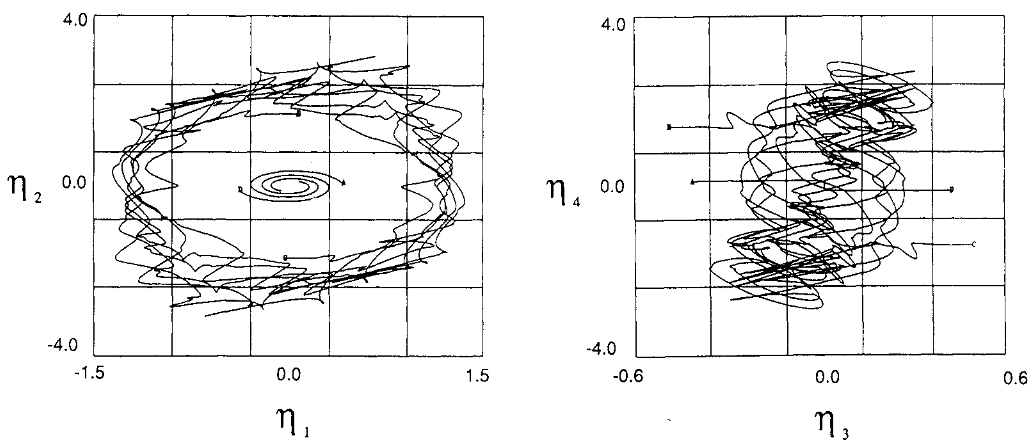

Fig. 7 Phase plane of nonlinear system for 12 orbits.

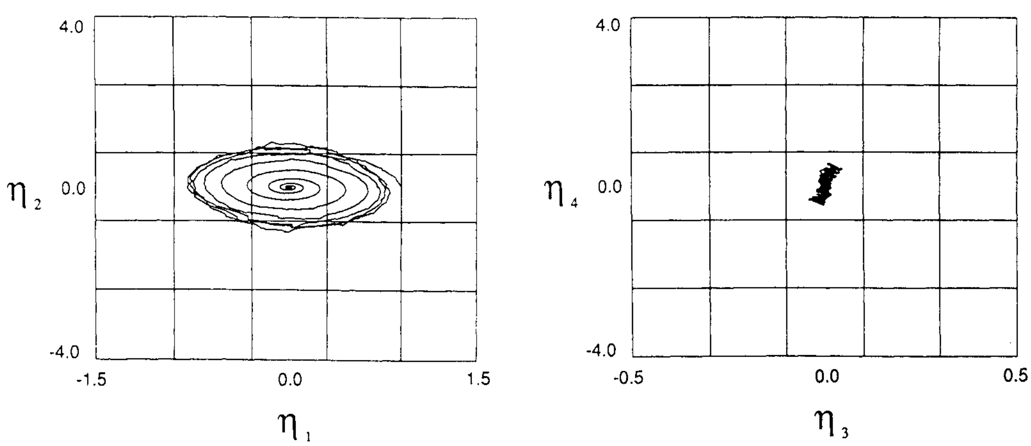

Fig. 8 Stable region of phase space.

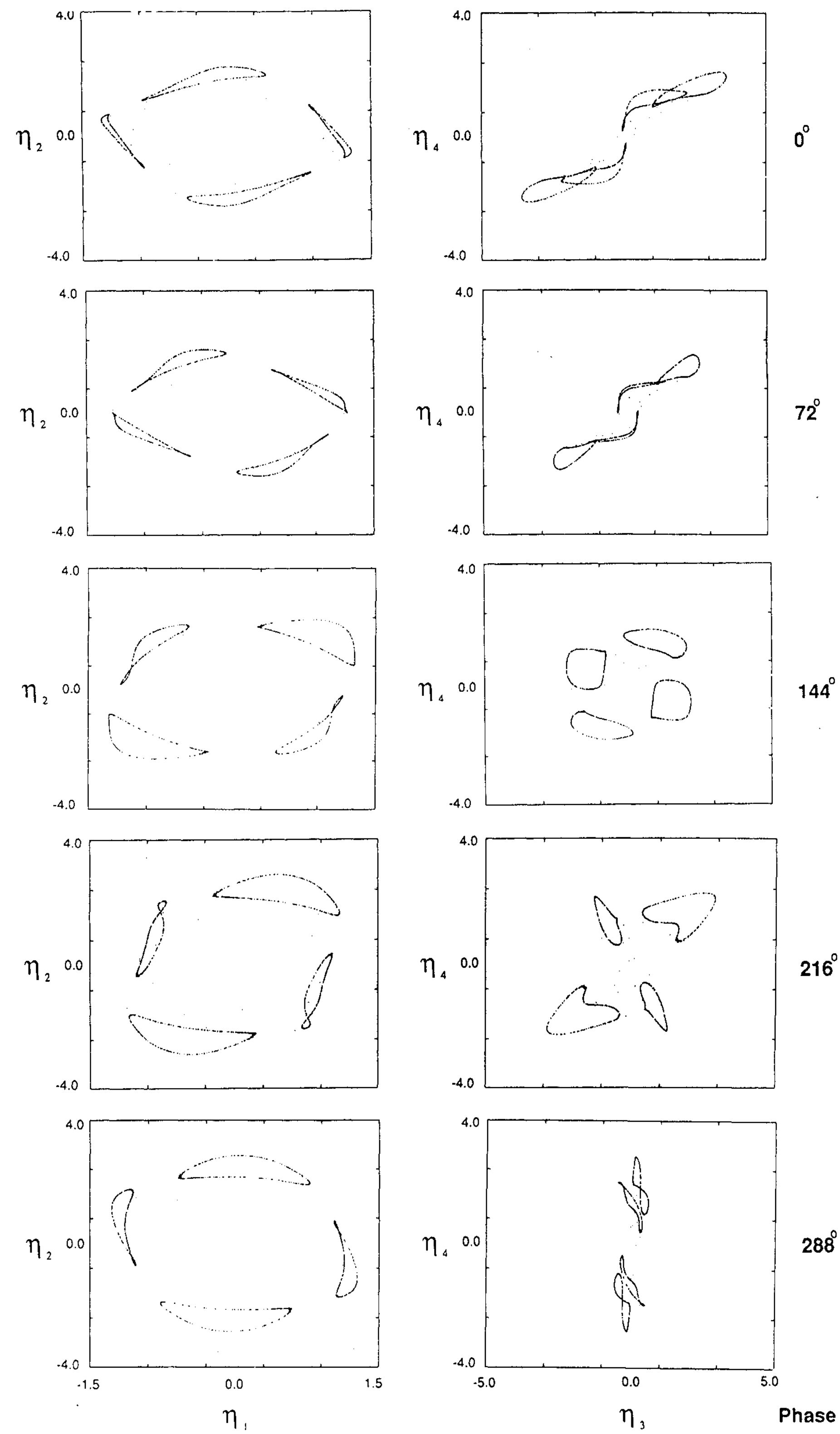

Fig. 9 Poincaré maps of nonlinear system behavior.

Substituting into Eq. (4) and premultiplying by $F^{-1}(\tau)$ yields

$$\bar{\eta}' = J\bar{\eta} + F^{-1}B\bar{u}(\tau) \tag{10}$$

The transformed system (10) has constant coefficients but a periodic control term. In Ref. 2, it is shown that a scalar control law of the form

$$\bar{u}(\tau) = [K_1, K_2, 0, 0]\bar{\eta}(\tau) \tag{11}$$

can always be found that will stabilize the system (4). In terms of the physical coordinates $\bar{x}(\tau)$, the control has the form

$$\bar{u}(\tau) = [K_1, K_2, 0, 0]F^{-1}(\tau)\bar{x}(\tau) \tag{12}$$

Several methods for choosing the gains K_1 and K_2 are developed in Ref. 2.

The controller design problem is to find the proper control gains K_1 and K_2, such that the Poincaré exponents are all negative. This stability applies only to the linearized system.

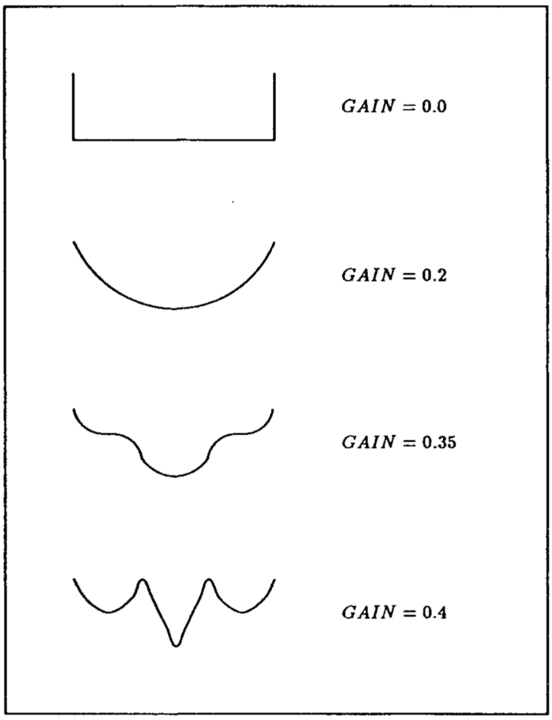

Fig. 10 Two-dimensional analogy.

The controller can be shown mathematically to be effective controlling the linearized system. To examine what happens when it is applied to the nonlinear system will require some special analysis techniques.

Analysis Methods

Several methods have been developed that can be applied to nonlinear systems. Often they do not provide the detailed design information of a linear analysis but show trends in the behavior and define the regions of stability.

Phase Space

The phase space used in this analysis is not typical. The space has been transformed into a set of modal coordinates and the magnitudes have been normalized relative to the orbit parameters as demonstrated in Ref. 1.

Poincaré Maps

A Poincaré map, also referred to as a Poincaré section, is constructed by defining a plane in three or higher dimensional phase space. Every time the trajectory passes through the plane, it is mapped as a point on the plane. The Poincaré maps can be categorized by the patterns that are generated.[5]

One method of generating Poincaré maps for a system that has a dominant frequency is to map the phase space onto a torus and take sections from the torus. The phase plane of the Poincaré map is perpendicular to the torus and cuts a cross section. The motion of a periodic system will make lines around the torus. These nonlinear methods are usually applied to three-dimensional or smaller systems. In this application we will adapt some of them to the four-dimensional system.

Results

We will examine some previous results using these techniques and extend the analysis into the nonlinear domain. First the uncontrolled system will be examined, for this is just the controlled system with the controller gain set to zero. Then the gain will be increased and the behavior of the linearized and nonlinear systems will be examined. All phase plane and Poincaré maps have been plotted in modal coordinates. The magnitudes shown are related to the physical perturbation but are only used to compare relative magnitude of the modes and general trends of behavior in this study.

Uncontrolled System

The uncontrolled system is a conservative Hamiltonian system. To examine the stability of this system, the angle between the spin axis of the satellite and the orbit normal was plotted. For 12 orbits there appears to be a definite increase in the angle from the original perturbation, indicating an unstable system. If the integration is continued for many orbits (Fig. 2), then the system appears to have a bounded behavior. This is what one should expect for a conservative system. If we use the Poincaré map, we again get a bounded behavior that is typical of conservative systems (Fig. 3).

The phase plane plots show what is occurring with the uncontrolled system; the modes are exchanging energy through the nonlinear terms. As two modes increase in magnitude the other two modes decrease. As two modes become saturated, the energy transfers to the other two modes (Fig. 4).

Controlled System

A controller, designed using the modal control technique for time-periodic systems, is added to damp the oscillations of the systems. To aid in setting the gain of the controller, we will first examine the linearized system via a plot of the Poincaré exponents (Fig. 5). As the gain is increased, the two unstable exponents move to the negative side of the real axis and become stable. When the gain is about 0.2, the exponents intersect the real axis. As the gain is increased there are two distinct real exponents. Classical control techniques can be used to optimize the system for the desired performance based on this plot. This will help us select the gain required for our controller, but it falls short in predicting the controller performance in the nonlinear system. If we examine the linearized system with the gain set at 0.4 in the modal phase space, the results are as expected for linear systems (Fig. 6). It should be noted that the coordinates η_i in the phase portraits in Fig. 7 are modal coordinates for the controlled system. As such no coupling exists between the pairs of coordinates.

The linearized system exhibits interesting behavior when the gain for the controller is such that the exponents are real and equal. The period of the system doubles. The uncontrolled system has a period of 2π, but when the gain reaches 0.2, the closed-loop system period doubles to 4π. This can be easily handled by analyzing the system as if it has a period of 4π even when the period is 2π.

Examining the nonlinear system in the modal phase space shows a behavior very similar to the linearized system near the equilibrium point, but further out, the system nonlinearities dominate as we see in Fig. 7. Exploring the four-dimensional phase space, one finds a region surrounding the equilibrium point that if perturbed to anywhere within that region, the controller will always damp out the motion (Fig. 8). Started outside this region, it moves into an orbital motion that almost appears random but yet on close examination seems to have some order. There is symmetry and bounded behavior. While for the nonlinear system the modal variables are coupled, they remain useful for displaying system behavior. To get a Poincaré map of this four-dimensional system, we will map the first two modes on one torus and the other two modes on a second torus.[6] This will allow us to examine the orbital-type behavior of this system. A period of 4π was selected for the torus, a logical choice from the analysis of the linear system. This will give two projections of the four-dimensional space. By using modal coordinates, the view of the system behavior is easier to analyze. The resulting map indicates a type of quasiperiodic behavior (Fig. 9). There are four regions on each map that appear to have been mapped onto the torus. By examining it over a full 4π cycle, one can see that the four regions are cyclic and have a period of 16π. The system now has a dominant period of 16π.

Where did this structure come from and how can we remove this undesirable feature from our controlled system? The formation of this structure can be explained with a two-dimensional analogy (Fig. 10). When the gain is low, the phase space is relatively flat. As the gain is increased, the phase space develops a bowl shape with the equilibrium point at the bottom, a stable system. As the gain increases further, the nonlinear terms develop a limit cycle type of structure around the equilibrium point. Further increases in gain result in the strange attractor that we have mapped. This behavior can be

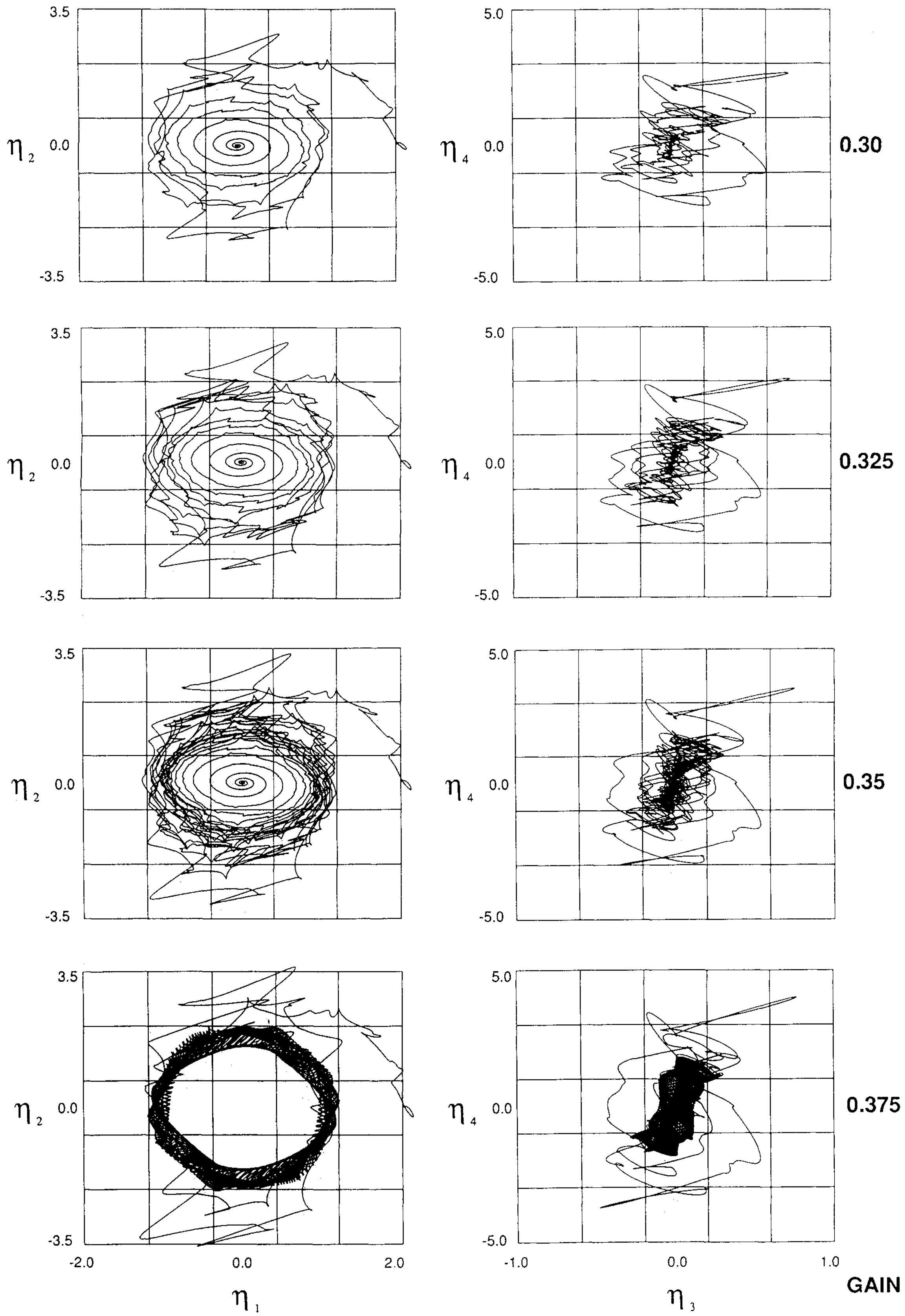

Fig. 11 Development of nonlinear structure.

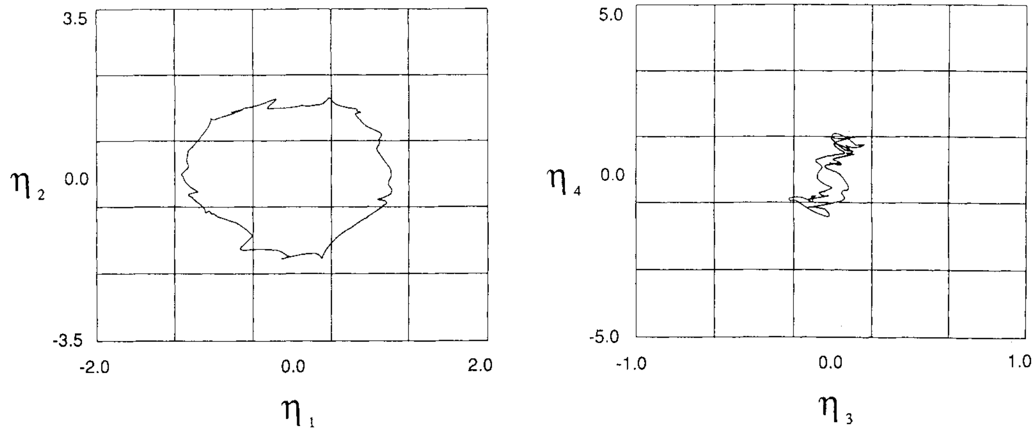

Fig. 12 Limit cycle structure.

observed in the phase plane plots of the system with various gain settings (Fig. 11). The limit cycle structure occurs with the gain set at 0.36 for this system (Fig. 12).

Conclusions

Controllers designed for linear time-periodic systems using a modal control scheme yield excellent results for predicting the small perturbation behavior of nonlinear systems. Based on the linear approximation, control gains can be set to match desired levels of system damping. However, when these controllers are evaluated for large motions, the nonlinear effects dominate the characteristics of the motion and the motion depends on controller gain in a more complex manner. It has been shown that for the system considered, at low gains, the origin is globally stable, but as gain increases, a critical value is reached, and above this value, stable periodic solutions exist. Solutions for small initial conditions still converge to the origin, but for larger initial conditions the solutions approach the stable periodic motions. A controller for the nonlinear system has to be designed to overcome the undesirable behavior.

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